EXERCISE 1

1) Implicit function thm for manifolds (Thm 2.8.16) would prove this point straight away. We need to find the appropriate diffeomorphism & show that M is the preimage of a regular value.

The definition of the set already gives away the right function.

Let $F: \mathbb{R}^4 \to \mathbb{R}^2$ be defined by

$$F(x_1, x_2, x_3, x_4) = (x_1^2 - x_4^2 - 1, x_2^2 + x_3^2 - 2)$$

F is polinomial in all variables, so clearly smooth.

Then $M \cap \mathbb{R}^4 = \{ \vec{x} \in \mathbb{R}^4 \mid F(\vec{x}) = (0,0) \}$, that is $M = F^{-1}(\vec{0})$ and its dimension (see also Proposition 2.8.16) is $4-2=0$. We still need to show that $(0,0)$ is a regular value for F.

For any $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ we have

$$dF_{(x_1,x_2,x_3,x_4)} = \begin{pmatrix} 2x_1 & 0 & 0 & -2x_4 \\ 0 & 2x_2 & 2x_3 & 0 \end{pmatrix}$$

On M, $F(\vec{x}) = \vec{0}$, so x_1 and x_4 cannot simultaneously vanish, and the same holds for x_2 and x_3

$\Rightarrow$ F has maximal rank, which means that $\vec{0}$ is a regular value concluding the exercise

2) By theorem 2.8.22, the tangent space can be computed as $\ker dF_p$.

$$\ker dF_{(1,-1,1,0)} = \ker \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 \end{pmatrix}$$

That is

$$T_{(1,-1,1,0)} M = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = 0,\ x_3 - x_2 = 0 \}$$

<u>Note</u> you can solve this exercise computing a local chart for the manifold, e.g.

$$\varphi(u,v) = \left(\sqrt{1+v^2},\ -\sqrt{2-u^2},\ u,\ v \right)$$

defined as a map from $(0,\sqrt{2})\times\mathbb{R}\to U$,
$U=\{\vec{x}\in\mathbb{R}^4 \mid x_1>0,\ -\sqrt{2}<x_2<0,\ 0<x_3<\sqrt{2}\}$, with inverse

$$\varphi^{-1}(x_1,x_2,x_3,x_4)=(x_3,x_4)$$

Then $d\varphi_{(1,0)}$ would give you the vectors that span the tangent space.

EXERCISE 2

1) Theorem 7.4.4 is the tool to compute pullbacks

$$\phi^*\omega = \phi^*(y)\,\phi^*(z)\,\phi^*(dx)\wedge\phi^*(dy)\wedge\phi^*(dz)$$
$$= (y\circ\phi)(z\circ\phi)\, d(x\circ\phi)\wedge d(y\circ\phi)\wedge d(z\circ\phi)$$

$$= 2uv\cdot t\cdot d(u^2-v^2)\wedge d(2uv)\wedge dt$$

$du\wedge du = dv\wedge dv = 0$

$$= 2uvt\,(2u\,du-2v\,dv)\wedge(2v\,du+2u\,dv)\wedge dt$$
$$= 8uvt\,(u^2\,du\wedge dv - v^2\,dv\wedge du)\wedge dt$$
$$= 8uvt\,(u^2+v^2)\,du\wedge dv\wedge dt$$

2) $\mathcal{L}_X\omega = i_X d\omega + d\, i_X\omega$

Since $\omega\in\Omega^3(\mathbb{R}^3)$ and $\Omega^4(\mathbb{R}^3)=\{0\}$ we know without need of computing anything that $d\omega=0$

$$\Rightarrow \mathcal{L}_X \omega = d\, \iota_X \omega$$

$$X = \frac{\partial}{\partial x} \Rightarrow \iota_X \omega = yz \underbrace{(\iota_X dx)}^{=1} \wedge dy \wedge dz$$

$$0 \equiv \begin{cases} + (-1)\, yz\, dx \wedge (\iota_X dy) \wedge dz \\ + yz\, dx \wedge dy \wedge (\iota_X dz) \end{cases}$$

$$= yz\, dy \wedge dz$$

$$\Rightarrow d\iota_X \omega = d(yz\, dy \wedge dz) = z\, dy \wedge dy \wedge dz + y\, dz \wedge dy \wedge dz$$

$= 0$ since $dy \wedge dy = dz \wedge dz = 0$

$$\Rightarrow \mathcal{L}_X \omega = 0$$

EXERCISE 3.

By definition $v^b := \langle v, \cdot \rangle$

For convenience denote $F : V \to V^*$

$$v \mapsto v^b = \langle v, \cdot \rangle$$

Since inner products are linear

$$F(\alpha X + \beta Y) = \langle \alpha X + \beta Y, \cdot \rangle = \alpha \langle X, \cdot \rangle + \beta \langle Y, \cdot \rangle$$

$$= \alpha F(X) + \beta F(Y)$$

That is, F and, thus, $\flat$ is linear.

We need to check that F is bijective. Since $\dim V = \dim V^*$, it is enough to check that F is injective (we did this multiple times in the course, see e.g. Remark 5.1.11).

By linearity, injectivity is equivalent to say $\mathrm{Ker}(F) = \{0\}$.

Let $X \in \mathrm{Ker}(F)$

$$\Rightarrow \forall Y \in V, \quad F(X)(Y) = \langle X, Y \rangle = 0$$

Pick $Y = X \Rightarrow F(X)(X) = 0$

$$F(X)(X) = \langle X, X \rangle = \|X\|^2$$

That is $X = 0$.

EXERCISE 4

1) $\pi : T^*M \longrightarrow M$ $\qquad (q,p) \in T^*M$ means $q \in M, p \in T^*_q M$

therefore $d\pi_{(q,p)} : T_{(q,p)}(T^*M) \longrightarrow T_q M$

and thus $d\pi^*_{(q,p)} : T^*_q M \longrightarrow T^*_{(q,p)}(T^*M)$

That is $\eta : T^*M \longrightarrow T^*(T^*M)$ and satisfies the section property by construction

$\Rightarrow \eta \in \Omega^1(T^*M)$

$\Pi : T^*(T^*M) \to T^*M$
projection to base
$\Rightarrow \Pi \circ \eta = id_{T^*M}$

2) Let (x^i, ξ_i) be local coords on T^*M
Since by definition $\pi(x,\xi) = x$,

$$d\pi^*(dx^i) = d(x^i \circ \pi) = dx^i$$

$$\Rightarrow \eta_{(x,\xi)} = d\pi^*(\xi_i dx^i) = \xi_i dx^i$$

3) If we show that there is a nowhere vanishing $2n$-form on T^*M, we are

done, since that is how we defined orientability.

First of all, $\omega = d\eta$ locally is $\omega = d\xi_i \wedge dx^i$ (by our previous point 2).

ω is just a two-form but is made of sums of all the $2n$ basis elements $\{dx^i, d\xi_i\}$, so taking n products of itself could yield a good $2n$-form. Indeed

$$\underbrace{\omega \wedge \dots \wedge \omega}_{n\text{-times}} = (d\xi_{i_1} \wedge dx^{i_1}) \wedge \dots \wedge (d\xi_{i_n} \wedge dx^{i_n})$$

Einstein sum explicited

$$\overset{\text{Einstein sum explicited}}{=} \sum_{\substack{1 \le i_1, \dots, i_n \le n \\ \text{all pairwise distinct}}} (d\xi_{i_1} \wedge dx^{i_1}) \wedge \dots \wedge (d\xi_{i_n} \wedge dx^{i_n})$$

$$= \sum_{\sigma \in S_n} (d\xi_{\sigma(1)} \wedge dx^{\sigma(1)}) \wedge \dots \wedge (d\xi_{\sigma(n)} \wedge dx^{\sigma(n)})$$

$$= n!\; d\xi_1 \wedge dx^1 \wedge \dots \wedge d\xi_n \wedge dx^n$$

there are $n!$ combination and the reordering of the terms, since they always come in pairs, has $+$ sign.

which is clearly a nowhere vanishing $2n$-form

EXERCISE 5

1) The integral of a n-form on a manifold is essentially the computation of the volume of a parallelotope pulled back from the manifold to the euclidean space via the atlas: If $(\mathcal{U}, \varphi)$ is a chart then

$\int_{\varphi(\mathcal{U})} (\varphi^{-1})^* \omega$ is effectively a Euclidean integral.

The relation to the volume of the parallelotope is apparent from the definition of the wedge product of 1-forms

$$(\omega^1 \wedge \dots \wedge \omega^n | v_1, \dots, v_n) = \det \begin{pmatrix} \omega^1(v_1) & \dots & \omega^1(v_n) \\ \vdots & & \\ \omega^n(v_1) & \dots & \omega^n(v_n) \end{pmatrix}$$

which directly corresponds to the signed volume of the parallelotope spanned by the vectors

$$X_i = (\omega^1(v_i), \dots, \omega^n(v_i))$$

A delicate point is how this pullback can be achieved, since the manifold is likely patched by charts. It turns out that partitions of unity don't jus allow us to decompose the integral over single charts in a natural way: the result

that one obtains turns out to be independent of the chosen partition of unity.

This boils down to the fact that partitions of unity sum up to 1 at each point, and thus they allow to transition from a covering to another essentially by exploiting a multiplication by 1 and resumming up.

One last technical remark is that we assumed n-forms to have compact support, this to avoid quircks (see Example 8.3.7) and to prevent convergence issues.

2) η is closed if $d\eta = 0$, here

$$d\eta = -dx^2 \wedge dx^1 + dx^1 \wedge dx^2 = 2 dx^1 \wedge dx^2 \neq 0$$

Since $H^1_{dR}(\mathbb{R}^2) = \{0\}$, η cannot be exact either.

3) $\int_{S^1} \eta = \int_{D^1} d\eta = 2\int_{D^1} dx^1 \wedge dx^2 = 2\pi$

$D^1 = \{\vec{x} \in \mathbb{R}^2 \mid \|x\| \leq 1\}$

Area of unit disk

<u>Note</u> alternatively you can compute it via the pullback by

$\varphi(x^1, x^2) = (\cos\theta, \sin\theta)$

$\Rightarrow \quad \varphi^* \eta = -\sin\theta \, d\cos\theta + \cos\theta \, d\sin\theta$
$= d\theta$

and thus $\int_{S^1} \eta = \int_0^{2\pi} d\theta = 2\pi$